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2009 J. Phys. A: Math. Theor. 42 115201

(http://iopscience.iop.org/1751-8121/42/11/115201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 115201 (17pp) doi:10.1088/1751-8113/42/11/115201

Lax forms of the q-Painlevé equations

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Received 8 September 2008, in final form 20 January 2009
Published 18 February 2009
Online at stacks.iop.org/JPhysA/42/115201

Abstract
All q-Painlevé equations which are obtained from the q-analog of the sixth Painlevé equation are expressed in a Lax formalism. They are characterized by the data of the associated linear q-difference equations. The degeneration pattern of the q-Painlevé equations is also presented.

PACS number: 02.30.Ik
Mathematics Subject Classification: 33E17, 34M55, 39A12

1. Introduction

Discrete Painlevé equations are studied from various points of view as integrable systems [5]. They are discrete equations which are reduced to the Painlevé equations in a suitable limiting process, and moreover, which pass the singularity confinement test. Passing this test can be thought of as a difference analog of the Painlevé property. The singularity confinement test has been proposed by Grammaticos *et al* as a criterion for the integrability of discrete dynamical systems [2].

Discrete Painlevé equations were classified on the basis of the types of rational surfaces connected to extended affine Weyl groups [7, 10]. There are three types of discrete Painlevé equations: elliptic-difference, q-difference and difference. The q-Painlevé equations are given by table 1. As is well-known, the sixth Painlevé equation yields the other five Painlevé equations by a process of coalescence. Among the q-Painlevé equations, the q-Painlevé equation of type A_0^* (q-$P(A_0^*)$) is the most generic one because the other q-Painlevé equations can be obtained from this equation by a limiting procedure. These equations are organized in the following degeneration pattern obtained through coalescence.

$$\begin{array}{ccccccccc} q\text{-}P(A_0^*) & \to & q\text{-}P(A_1) & \to & q\text{-}P(A_2) & \to & q\text{-}P(A_3) & \to & \\ \to\ q\text{-}P(A_4) & \to & q\text{-}P(A_5) & \to & q\text{-}P(A_6) & \to & q\text{-}P(A_7) & & \\ & \searrow & & \nearrow & & \nearrow & & & \\ & & q\text{-}P(A_5)^\sharp & \to & q\text{-}P(A_6)^\sharp & \to & q\text{-}P(A_7'). & & \end{array}$$

Table 1. The q-Painlevé equations.

Abbrev.	$q\text{-}P(A_0^*)$	$q\text{-}P(A_1)$	$q\text{-}P(A_2)$	$q\text{-}P(A_3)$	$q\text{-}P(A_4)$
Surface	$A_0^{(1)*}$	$A_1^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$A_4^{(1)}$
Symmetry	$E_8^{(1)}$	$E_7^{(1)}$	$E_6^{(1)}$	$D_5^{(1)}$	$A_4^{(1)}$
$q\text{-}P(A_5)$	$q\text{-}P(A_5)^\sharp$	$q\text{-}P(A_6)$	$q\text{-}P(A_6)^\sharp$	$q\text{-}P(A_7)$	$q\text{-}P(A_7')$
$A_5^{(1)}$	$A_5^{(1)}$	$A_6^{(1)}$	$A_6^{(1)}$	$A_7^{(1)}$	$A_7^{(1)\prime}$
$(A_2+A_1)^{(1)}$	$(A_2+A_1)^{(1)}$	$(A_1+A_1)^{(1)}$	$(A_1+A_1)^{(1)}$	$A_1^{(1)}$	$A_1^{(1)}$

Another important aspect of the Painlevé equations is their connection to the monodromy-preserving deformation of linear differential equations. The generalized Riemann problem was already studied for linear differential, difference and q-difference equations in the Birkhoff's paper, [1]. Jimbo and Sakai studied the deformation of a 2×2 matrix system of q-difference equations and found the q-Painlevé equation of type A_3 ($q\text{-}P(A_3)$), which is commonly known as $q\text{-}P_{VI}$ [4]. Sakai also found a Lax form of the q-Painlevé equation of type A_2 ($q\text{-}P(A_2)$), a particular case of a q-Garnier system [8, 9]. Hay *et al* found Lax forms of the q-Painlevé equations, reductions from a Lax pair for a lattice modified KdV equation [3]. However, Lax forms of a lot of q-Painlevé equations have not been obtained yet.

In this paper, we present Lax pairs of all q-Painlevé equations which are obtained from $q\text{-}P(A_3)$. In section 2, we illustrate the connection preserving deformation and derive $q\text{-}P(A_3)$. We also propose Lax pairs of $q\text{-}P(A_4)$, $q\text{-}P(A_5)$, $q\text{-}P(A_5)^\sharp$, $q\text{-}P(A_6)$, $q\text{-}P(A_6)^\sharp$, $q\text{-}P(A_7)$ and $q\text{-}P(A_7')$. In section 3, we give replacements of the parameters for the degeneration. The Lax form of $q\text{-}P(A_3)$ can be obtained from the Lax form of $q\text{-}P(A_2)$. In section 4, we give the Lax form of $q\text{-}P(A_2)$ and replacements of the parameters for the degeneration.

2. Lax forms of q-Painlevé equations

2.1. Derivation of $q\text{-}P(A_3)$

In this subsection, we illustrate the connection preserving deformation and derive the q-Painlevé equation of type A_3 in the paper, [4].

Consider a 2×2 matrix system with polynomial coefficients

$$Y(qx, t) = A(x, t)Y(x, t). \tag{1}$$

The connection preserving deformation of the linear q-difference equation, which is a discrete counterpart of monodromy preserving deformation, is equivalent to existence of a linear deformation equation whose coefficients are rational in x. We express the deformation equation in the form

$$Y(x, qt) = B(x, t)Y(x, t). \tag{2}$$

The compatibility condition for the systems (1) and (2) reads

$$A(x, qt)B(x, t) = B(qx, t)A(x, t). \tag{3}$$

$q\text{-}P(A_3)$ can be obtained from the condition (3). We take $A(x, t)$ to be of the form

$$A(x, t) = A_0(t) + xA_1(t) + x^2A_2, \tag{4}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, \theta_2 t, \tag{5}$$

$$\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3)(x-a_4). \tag{6}$$

Here the parameters κ_j, θ_j, a_j are independent of t. We have

$$\kappa_1\kappa_2 a_1a_2a_3a_4 = \theta_1\theta_2.$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y,t) = 0, \qquad A_{11}(y,t) = \kappa_1 z_1, \qquad A_{22}(y,t) = \kappa_2 z_2, \tag{7}$$

so that

$$z_1z_2 = \kappa_1\kappa_2(y-a_1t)(y-a_2t)(y-a_3)(y-a_4).$$

The matrix $A(x,t)$ can be parametrized as

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & \kappa_2 w(x-y) \\ \kappa_1 w^{-1}(\gamma x+\delta) & \kappa_2((x-y)(x-\beta)+z_2) \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1-\kappa_2}[y^{-1}((\theta_1+\theta_2)t - \kappa_1z_1 - \kappa_2z_2) - \kappa_2((a_1+a_2)t + a_3 + a_4 - 2y)],$$
$$\beta = \frac{1}{\kappa_1-\kappa_2}[-y^{-1}((\theta_1+\theta_2)t - \kappa_1z_1 - \kappa_2z_2) + \kappa_1((a_1+a_2)t + a_3 + a_4 - 2y)],$$
$$\gamma = z_1 + z_2 + (y+\alpha)(y+\beta) + (\alpha+\beta)y - a_1a_2t^2 - (a_1+a_2)(a_3+a_4)t - a_3a_4,$$
$$\delta = y^{-1}(a_1a_2a_3a_4t^2 - (\alpha y + z_1)(\beta y + z_2)).$$

The quantity $w = w(t)$ is related to the 'gauge' freedom, and does not enter the final result for the q-$P(A_3)$. The matrix $B(x,t)$ is a rational function of the form

$$B(x,t) = \frac{x}{(x-a_1qt)(x-a_2qt)}(xI + B_0(t)). \tag{8}$$

The compatibility (3) is equivalent to

$$A(a_iqt, qt)(a_iqtI + B_0(t)) = 0 \quad (i = 1, 2), \tag{9}$$
$$(a_iqtI + B_0(t))A(a_it, t) = 0 \quad (i = 1, 2), \tag{10}$$
$$A_0(qt)B_0(t) = qB_0(t)A_0(t). \tag{11}$$

Substituting the parametrization above, one obtains a set of q-difference equations. Let us use the notations $\bar{y} = y(qt)$ and so forth. Introduce z by

$$z_1 = \frac{(y-a_1t)(y-a_2t)}{\kappa_1qz}, \qquad z_2 = \kappa_1qz(y-a_3)(y-a_4).$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} = \frac{-\kappa_2q\bar{z}}{1-\kappa_2\bar{z}}\left(-\beta + \frac{t(a_1+a_2)-y}{\kappa_2\bar{z}}\right),$$
$$B_{12} = \frac{\kappa_2qw\bar{z}}{1-\kappa_2\bar{z}},$$
$$B_{21} = \frac{\kappa_1q\bar{z}}{w(1-\kappa_1q\bar{z})}\left(a_1qt - \bar{\alpha} + \frac{a_2qt-\bar{y}}{\kappa_1q\bar{z}}\right)\left(a_1t - \beta + \frac{a_2t-y}{\kappa_2\bar{z}}\right)$$
$$= \frac{\kappa_1q\bar{z}}{w(1-\kappa_1q\bar{z})}\left(a_2qt - \bar{\alpha} + \frac{a_1qt-\bar{y}}{\kappa_1q\bar{z}}\right)\left(a_2t - \beta + \frac{a_1t-y}{\kappa_2\bar{z}}\right),$$
$$B_{22} = \frac{-\kappa_1q\bar{z}}{1-\kappa_1q\bar{z}}\left(-\bar{\alpha} + \frac{qt(a_1+a_2)-\bar{y}}{\kappa_1q\bar{z}}\right).$$

Set further

$$b_1 = \frac{a_1a_2}{\theta_1}, \qquad b_2 = \frac{a_1a_2}{\theta_2}, \qquad b_3 = \frac{1}{\kappa_1 q}, \qquad b_4 = \frac{1}{\kappa_2}. \tag{12}$$

The equations (9)–(11) are equivalent to

$$\frac{y\bar{y}}{a_3a_4} = \frac{(\bar{z}-b_1t)(\bar{z}-b_2t)}{(\bar{z}-b_3)(\bar{z}-b_4)}, \tag{13}$$

$$\frac{z\bar{z}}{b_3b_4} = \frac{(y-a_1t)(y-a_2t)}{(y-a_3)(y-a_4)}, \tag{14}$$

$$\frac{\bar{w}}{w} = \frac{b_4}{b_3}\frac{\bar{z}-b_3}{\bar{z}-b_4}. \tag{15}$$

We have a single constraint

$$\frac{b_1b_2}{b_3b_4} = q\frac{a_1a_2}{a_3a_4}.$$

q-$P(A_3)$ is (13) and (14).

2.2. *Lax form of q-$P(A_4)$*

A Lax pair for the q-Painlevé equation of type A_4 is a linear probrem of the form

$$Y(qx,t) = A(x,t)Y(x,t), \tag{16}$$

$$Y(x,qt) = B(x,t)Y(x,t), \tag{17}$$

whose compatibility condition, namely,

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{18}$$

can express q-$P(A_4)$. We take $A(x,t)$ to be of the form

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2, \tag{19}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1t, \theta_2t, \tag{20}$$

$$\det A(x,t) = \kappa_1\kappa_2(x-a_1t)(x-a_2t)(x-a_3). \tag{21}$$

We have

$$-\kappa_1\kappa_2a_1a_2a_3 = \theta_1\theta_2.$$

The matrix $B(x,t)$ is a rational function of the form

$$B(x,t) = \frac{x}{(x-a_1qt)(x-a_2qt)}(xI + B_0(t)). \tag{22}$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y,t) = 0, \qquad A_{11}(y,t) = \kappa_1z_1, \qquad A_{22}(y,t) = z_2, \tag{23}$$

so that

$$z_1z_2 = \kappa_2(y-a_1t)(y-a_2t)(y-a_3).$$

The matrix $A(x,t)$ can be parametrized as

$$A(x,t)=\begin{pmatrix}\kappa_1((x-y)(x-\alpha)+z_1) & w(x-y)\\ \kappa_1 w^{-1}(\gamma x+\delta) & \kappa_2(x-y)+z_2\end{pmatrix}.$$

Here

$$\alpha=\frac{1}{\kappa_1}[y^{-1}((\theta_1+\theta_2)t-\kappa_1z_1-z_2)+\kappa_2],$$
$$\gamma=z_2-\kappa_2((2y+\alpha)-(a_1+a_2)t-a_3),$$
$$\delta=y^{-1}(-\kappa_2a_1a_2a_3t^2-(\alpha y+z_1)(-\kappa_2y+z_2)).$$

Introduce z by

$$z_1=\frac{(y-a_1t)(y-a_2t)}{\kappa_1qz},\qquad z_2=\kappa_1\kappa_2qz(y-a_3).$$

Then the matrix $B_0(t)=(B_{ij})$ is parametrized as follows:

$$B_{11}=-q\bar{z}\left(\kappa_2+\frac{t(a_1+a_2)-y}{\bar{z}}\right),$$
$$B_{12}=qw\bar{z},$$
$$B_{21}=\frac{\kappa_1q\bar{z}}{w(1-\kappa_1q\bar{z})}\left(a_1qt-\bar{\alpha}+\frac{a_2qt-\bar{y}}{\kappa_1q\bar{z}}\right)\left(\kappa_2+\frac{a_2t-y}{\bar{z}}\right)$$
$$=\frac{\kappa_1q\bar{z}}{w(1-\kappa_1q\bar{z})}\left(a_2qt-\bar{\alpha}+\frac{a_1qt-\bar{y}}{\kappa_1q\bar{z}}\right)\left(\kappa_2+\frac{a_1t-y}{\bar{z}}\right),$$
$$B_{22}=\frac{-\kappa_1q\bar{z}}{1-\kappa_1q\bar{z}}\left(-\bar{\alpha}+\frac{qt(a_1+a_2)-\bar{y}}{\kappa_1q\bar{z}}\right).$$

Set further

$$b_1=\frac{a_1a_2}{\theta_1},\qquad b_2=\frac{a_1a_2}{\theta_2},\qquad b_3=\frac{1}{\kappa_1q},\qquad a_4=-\kappa_2. \tag{24}$$

The compatibility (18) is equivalent to

$$\frac{y\bar{y}}{a_3a_4}=-\frac{(\bar{z}-b_1t)(\bar{z}-b_2t)}{\bar{z}-b_3}, \tag{25}$$
$$\frac{z\bar{z}}{b_3}=-\frac{(y-a_1t)(y-a_2t)}{a_4(y-a_3)}, \tag{26}$$
$$\frac{\bar{w}}{w}=-\frac{\bar{z}}{b_3}+1. \tag{27}$$

We have a constraint

$$\frac{b_1b_2}{b_3}=q\frac{a_1a_2}{a_3a_4}.$$

q-$P(A_4)$ is (25) and (26).

2.3. Lax form of q-$P(A_5)$

A Lax pair for the q-Painlevé equation of type A_5 is a linear probrem of the form

$$Y(qx,t)=A(x,t)Y(x,t), \tag{28}$$

$$Y(x, qt) = B(x, t)Y(x, t), \tag{29}$$

whose compatibility condition, namely,

$$A(x, qt)B(x, t) = B(qx, t)A(x, t) \tag{30}$$

can express q-$P(A_5)$. We take $A(x, t)$ to be of the form

$$A(x, t) = A_0(t) + xA_1(t) + x^2 A_2, \tag{31}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, 0, \tag{32}$$

$$\det A(x, t) = \kappa_1\kappa_2 x(x - a_1 t)(x - a_2 t). \tag{33}$$

The matrix $B(x, t)$ is a rational function of the form

$$B(x, t) = \frac{x}{(x - a_1 qt)(x - a_2 qt)}(xI + B_0(t)). \tag{34}$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y, t) = 0, \qquad A_{11}(y, t) = \kappa_1 z_1, \qquad A_{22}(y, t) = z_2, \tag{35}$$

so that

$$z_1 z_2 = \kappa_2 y(y - a_1 t)(y - a_2 t).$$

The matrix $A(x, t)$ can be parametrized as

$$A(x, t) = \begin{pmatrix} \kappa_1((x - y)(x - \alpha) + z_1) & w(x - y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2(x - y) + z_2 \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1}[y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2) + \kappa_2],$$
$$\gamma = z_2 - \kappa_2(2y + \alpha - t(a_1 + a_2)),$$
$$\delta = -y^{-1}(\alpha y + z_1)(-\kappa_2 y + z_2).$$

Introduce z by

$$z_1 = \frac{(y - a_1 t)(y - a_2 t)}{\kappa_1 q z}, \qquad z_2 = \kappa_1\kappa_2 qzy.$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} = -q\bar{z}\left(\kappa_2 + \frac{t(a_1 + a_2) - y}{\bar{z}}\right),$$
$$B_{12} = qw\bar{z},$$
$$B_{21} = \frac{\kappa_1 q\bar{z}}{w(1 - \kappa_1 q\bar{z})}\left(a_1 qt - \bar{\alpha} + \frac{a_2 qt - \bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2 + \frac{a_2 t - y}{\bar{z}}\right)$$
$$= \frac{\kappa_1 q\bar{z}}{w(1 - \kappa_1 q\bar{z})}\left(a_2 qt - \bar{\alpha} + \frac{a_1 qt - \bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2 + \frac{a_1 t - y}{\bar{z}}\right),$$
$$B_{22} = \frac{-\kappa_1 q\bar{z}}{1 - \kappa_1 q\bar{z}}\left(-\bar{\alpha} + \frac{qt(a_1 + a_2) - \bar{y}}{\kappa_1 q\bar{z}}\right).$$

Set further

$$b_1 = \frac{a_1 a_2}{\theta_1}, \qquad b_2 = -\frac{\theta_1}{\kappa_1\kappa_2}, \qquad b_3 = \frac{1}{\kappa_1 q}, \qquad a_4 = -\kappa_2. \tag{36}$$

The compatibility (30) is equivalent to

$$\frac{y\bar{y}}{a_4} = \frac{b_2 t(\bar{z} - b_1 t)}{\bar{z} - b_3}, \tag{37}$$

$$\frac{z\bar{z}}{b_3} = -\frac{(y - a_1 t)(y - a_2 t)}{a_4 y}, \tag{38}$$

$$\frac{\bar{w}}{w} = -\frac{\bar{z}}{b_3} + 1. \tag{39}$$

We have a constraint

$$\frac{b_1 b_2}{b_3} = q\frac{a_1 a_2}{a_4}.$$

q-$P(A_5)$ is (37) and (38).

2.4. Lax form of q-$P(A_5)^\sharp$

A Lax pair for the q-Painlevé equation of type $A_5^\sharp$ is a linear probrem of the form

$$Y(qx, t) = A(x, t)Y(x, t), \tag{40}$$

$$Y(x, qt) = B(x, t)Y(x, t), \tag{41}$$

whose compatibility condition, namely,

$$A(x, qt)B(x, t) = B(qx, t)A(x, t) \tag{42}$$

can express q-$P(A_5)^\sharp$. We take $A(x, t)$ to be of the form

$$A(x, t) = A_0(t) + xA_1(t) + x^2 A_2, \tag{43}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, 0, \tag{44}$$

$$\det A(x, t) = \kappa_1\kappa_2 x(x - a_1 t)(x - a_3). \tag{45}$$

The matrix $B(x, t)$ is a rational function of the form

$$B(x, t) = \frac{1}{x - a_1 qt}(xI + B_0(t)). \tag{46}$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y, t) = 0, \qquad A_{11}(y, t) = \kappa_1 z_1, \qquad A_{22}(y, t) = z_2, \tag{47}$$

so that

$$z_1 z_2 = \kappa_2 y(y - a_1 t)(y - a_3).$$

The matrix $A(x, t)$ can be parametrized as

$$A(x, t) = \begin{pmatrix} \kappa_1((x - y)(x - \alpha) + z_1) & w(x - y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2(x - y) + z_2 \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1}[y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2) + \kappa_2],$$

$$\gamma = z_2 - \kappa_2(2y + \alpha - a_1 t - a_3),$$

$$\delta = -y^{-1}(\alpha y + z_1)(-\kappa_2 y + z_2).$$

Introduce z by

$$z_1 = \frac{y(y-a_1t)}{\kappa_1 q z}, \qquad z_2 = \kappa_1\kappa_2 q z(y-a_3).$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$\begin{aligned}
B_{11} &= -q\bar{z}\left(\kappa_2 + \frac{a_1t-y}{\bar{z}}\right),\\
B_{12} &= qw\bar{z},\\
B_{21} &= \frac{\kappa_1 q\bar{z}}{w(1-\kappa_1 q\bar{z})}\left(a_1qt-\bar{\alpha}-\frac{\bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2-\frac{y}{\bar{z}}\right)\\
&= \frac{\kappa_1 q\bar{z}}{w(1-\kappa_1 q\bar{z})}\left(-\bar{\alpha}+\frac{a_1qt-\bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2+\frac{a_1t-y}{\bar{z}}\right),\\
B_{22} &= \frac{-\kappa_1 q\bar{z}}{1-\kappa_1 q\bar{z}}\left(-\bar{\alpha}+\frac{a_1qt-\bar{y}}{\kappa_1 q\bar{z}}\right).
\end{aligned}$$

Set further

$$b_1 = \frac{a_1}{\theta_1}, \qquad b_2 = -\frac{\theta_1}{\kappa_1\kappa_2 a_3}, \qquad b_3 = \frac{1}{\kappa_1 q}, \qquad a_4 = -\kappa_2. \tag{48}$$

The compatibility (42) is equivalent to

$$\frac{y\bar{y}}{a_3a_4} = -\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3}, \tag{49}$$

$$\frac{z\bar{z}}{b_3} = -\frac{y(y-a_1t)}{a_4(y-a_3)}, \tag{50}$$

$$\frac{\bar{w}}{w} = -\frac{\bar{z}}{b_3}+1. \tag{51}$$

We have a constraint

$$\frac{b_1b_2}{b_3} = q\frac{a_1}{a_3a_4}.$$

q-$P(A_5)^\sharp$ is (49) and (50).

2.5. *Lax form of q-$P(A_6)$*

A Lax pair for the q-Painlevé equation of type A_6 is a linear probrem of the form

$$Y(qx,t) = A(x,t)Y(x,t), \tag{52}$$

$$Y(x,qt) = B(x,t)Y(x,t), \tag{53}$$

whose compatibility condition, namely,

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{54}$$

can express q-$P(A_6)$. We take $A(x,t)$ to be of the form

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2, \tag{55}$$

$$A_2 = \begin{pmatrix}\kappa_1 & 0\\ 0 & 0\end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, 0, \tag{56}$$

$$\det A(x,t) = \kappa_1\kappa_2 x^2(x-a_1t). \tag{57}$$

The matrix $B(x,t)$ is a rational function of the form

$$B(x,t) = \frac{1}{x-a_1qt}(xI + B_0(t)). \tag{58}$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y,t) = 0, \qquad A_{11}(y,t) = \kappa_1 z_1, \qquad A_{22}(y,t) = z_2, \tag{59}$$

so that

$$z_1z_2 = \kappa_2 y^2(y-a_1t).$$

The matrix $A(x,t)$ can be parametrized as

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & w(x-y) \\ \kappa_1 w^{-1}(\gamma x+\delta) & \kappa_2(x-y)+z_2 \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1}[y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2) + \kappa_2],$$
$$\gamma = z_2 - \kappa_2(2y+\alpha-a_1t),$$
$$\delta = -y^{-1}(\alpha y + z_1)(-\kappa_2 y + z_2).$$

Introduce z by

$$z_1 = \frac{y(y-a_1t)}{\kappa_1 qz}, \qquad z_2 = \kappa_1\kappa_2 qzy.$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} = -q\bar{z}\left(\kappa_2 + \frac{a_1t-y}{\bar{z}}\right),$$
$$B_{12} = qw\bar{z},$$
$$B_{21} = \frac{\kappa_1 q\bar{z}}{w(1-\kappa_1 q\bar{z})}\left(a_1qt - \bar{\alpha} - \frac{\bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2 - \frac{y}{\bar{z}}\right)$$
$$= \frac{\kappa_1 q\bar{z}}{w(1-\kappa_1 q\bar{z})}\left(-\bar{\alpha} + \frac{a_1qt-\bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2 + \frac{a_1t-y}{\bar{z}}\right).$$
$$B_{22} = \frac{-\kappa_1 q\bar{z}}{1-\kappa_1 q\bar{z}}\left(-\bar{\alpha} + \frac{a_1qt-\bar{y}}{\kappa_1 q\bar{z}}\right).$$

Set further

$$b_1 = \frac{a_1}{\theta_1}, \qquad b_2 = -\frac{\theta_1}{\kappa_1\kappa_2}, \qquad b_3 = \frac{1}{\kappa_1 q}, \qquad a_4 = -\kappa_2. \tag{60}$$

The compatibility (54) is equivalent to

$$\frac{y\bar{y}}{a_4} = \frac{b_2t\bar{z}}{\bar{z}-b_3}, \tag{61}$$

$$\frac{z\bar{z}}{b_3} = -\frac{y-a_1t}{a_4}, \tag{62}$$

$$\frac{\bar{w}}{w} = -\frac{\bar{z}}{b_3} + 1. \tag{63}$$

We have a constraint

$$\frac{b_1b_2}{b_3} = q\frac{a_1}{a_4}.$$

q-$P(A_6)$ is (61) and (62).

2.6. Lax form of q-$P(A_6)^\sharp$

A Lax pair for the q-Painlevé equation of type $A_6^\sharp$ is a linear probrem of the form

$$Y(qx,t) = A(x,t)Y(x,t), \tag{64}$$

$$Y(x,qt) = B(x,t)Y(x,t), \tag{65}$$

whose compatibility condition, namely,

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{66}$$

can express q-$P(A_6)^\sharp$. We take $A(x,t)$ to be of the form

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2, \tag{67}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, 0, \tag{68}$$

$$\det A(x,t) = \kappa_1\kappa_2 x^2(x-a_3), \tag{69}$$

The matrix $B(x,t)$ is a rational function of the form

$$B(x,t) = \frac{1}{x}(xI + B_0(t)). \tag{70}$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1,2)$ by

$$A_{12}(y,t) = 0, \qquad A_{11}(y,t) = \kappa_1 z_1, \qquad A_{22}(y,t) = z_2, \tag{71}$$

so that

$$z_1 z_2 = \kappa_2 y^2(y-a_3).$$

The matrix $A(x,t)$ can be parametrized as

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & w(x-y) \\ \kappa_1 w^{-1}(\gamma x+\delta) & \kappa_2(x-y)+z_2 \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1}[y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2) + \kappa_2],$$
$$\gamma = z_2 - \kappa_2(2y + \alpha - a_3),$$
$$\delta = -y^{-1}(\alpha y + z_1)(-\kappa_2 y + z_2).$$

Introduce z by

$$z_1 = \frac{y^2}{\kappa_1 q z}, \qquad z_2 = \kappa_1\kappa_2 q z(y - a_3).$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} = -q\bar{z}\left(\kappa_2 - \frac{y}{\kappa_2\bar{z}}\right),$$
$$B_{12} = qw\bar{z},$$
$$B_{21} = \frac{\kappa_1 q}{w(1-\kappa_1 q\bar{z})}\left(-\bar{\alpha} - \frac{\bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2 - \frac{y}{\bar{z}}\right),$$
$$B_{22} = \frac{-\kappa_1 q\bar{z}}{1-\kappa_1 q\bar{z}}\left(-\bar{\alpha} - \frac{\bar{y}}{\kappa_1 q\bar{z}}\right).$$

Set further

$$b_1 = \frac{1}{\theta_1}, \qquad b_2 = -\frac{\theta_1}{\kappa_1\kappa_2 a_3}, \qquad b_3 = \frac{1}{\kappa_1 q}, \qquad a_4 = -\kappa_2. \tag{72}$$

The compatibility (66) is equivalent to

$$\frac{y\bar{y}}{a_3a_4} = -\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3}, \tag{73}$$

$$\frac{z\bar{z}}{b_3} = -\frac{y^2}{a_4(y-a_3)}, \tag{74}$$

$$\frac{\bar{w}}{w} = -\frac{\bar{z}}{b_3} + 1. \tag{75}$$

We have a constraint

$$\frac{b_1b_2}{b_3} = q\frac{1}{a_3a_4}.$$

q-$P(A_6)^\sharp$ is (73) and (74).

2.7. *Lax form of* q-$P(A_7)$

A Lax pair for the q-Painlevé equation of type A_7 is a linear probrem of the form

$$Y(qx,t) = A(x,t)Y(x,t), \tag{76}$$

$$Y(x,qt) = B(x,t)Y(x,t), \tag{77}$$

whose compatibility condition, namely,

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{78}$$

can express q-$P(A_7)$. We take $A(x,t)$ to be of the form

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2, \tag{79}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, 0, \tag{80}$$

$$\det A(x,t) = \kappa_1\kappa_2 x^3. \tag{81}$$

The matrix $B(x,t)$ is a rational function of the form

$$B(x,t) = \frac{1}{x}(xI + B_0(t)). \tag{82}$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y,t) = 0, \qquad A_{11}(y,t) = \kappa_1 z_1, \qquad A_{22}(y,t) = z_2, \tag{83}$$

so that

$$z_1z_2 = \kappa_2 y^3.$$

The matrix $A(x,t)$ can be parametrized as

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & w(x-y) \\ \kappa_1 w^{-1}(\gamma x+\delta) & \kappa_2(x-y)+z_2 \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1}[y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2) + \kappa_2],$$
$$\gamma = z_2 - \kappa_2(2y + \alpha),$$
$$\delta = -y^{-1}(\alpha y + z_1)(-\kappa y + z_2).$$

Introduce z by

$$z_1 = \frac{y^2}{\kappa_1 q z}, \qquad z_2 = \kappa_1\kappa_2 q y z.$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} = -q\bar{z}\left(\kappa_2 - \frac{y}{\bar{z}}\right),$$
$$B_{12} = q w \bar{z},$$
$$B_{21} = \frac{\kappa_1 q\bar{z}}{w(1-\kappa_1 q\bar{z})}\left(-\bar{\alpha} - \frac{\bar{y}}{\kappa_1 q\bar{z}}\right)\left(\kappa_2 - \frac{y}{\bar{z}}\right),$$
$$B_{22} = \frac{-\kappa_1 q\bar{z}}{1-\kappa_1 q\bar{z}}\left(-\bar{\alpha} - \frac{\bar{y}}{\kappa_1 q\bar{z}}\right).$$

Set further

$$b_1 = \frac{1}{\theta_1}, \qquad b_2 = -\frac{\theta_1}{\kappa_1\kappa_2}, \qquad b_3 = \frac{1}{\kappa_1 q}, \qquad a_4 = -\kappa_2. \tag{84}$$

The compatibility (78) is equivalent to

$$\frac{y\bar{y}}{a_4} = \frac{b_2 t\bar{z}}{\bar{z} - b_3}, \tag{85}$$

$$\frac{z\bar{z}}{b_3} = -\frac{y}{a_4}, \tag{86}$$

$$\frac{\bar{w}}{w} = -\frac{\bar{z}}{b_3} + 1. \tag{87}$$

We have a constraint

$$\frac{b_1 b_2}{b_3} = q\frac{1}{a_4}.$$

q-$P(A_7)$ is (85) and (86).

2.8. *Lax form of* q-$P(A_7')$

A Lax pair for the q-Painlevé equation of type A_7' is a linear probrem of the form

$$Y(qx,t) = A(x,t)Y(x,t), \tag{88}$$

$$Y(x,qt) = B(x,t)Y(x,t), \tag{89}$$

whose compatibility condition, namely,

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{90}$$

can express q-$P(A_7')$. We take $A(x,t)$ to be of the form

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2, \tag{91}$$

$$A_2 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, 0, \tag{92}$$

$$\det A(x,t) = \kappa_1\kappa_2 x^2. \tag{93}$$

The matrix $B(x,t)$ is a rational function of the form

$$B(x,t) = \frac{1}{x}(xI + B_0(t)). \tag{94}$$

Define $y = y(t)$, $z_i = z_i(t)$ $(i = 1, 2)$ by

$$A_{12}(y,t) = 0, \qquad A_{11}(y,t) = \kappa_1 z_1, \qquad A_{22}(y,t) = z_2, \tag{95}$$

so that

$$z_1 z_2 = \kappa_2 y^2.$$

The matrix $A(x,t)$ can be parametrized as

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & w(x-y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & z_2 \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1} y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2),$$
$$\gamma = z_2 + \kappa_2,$$
$$\delta = -y^{-1} z_2(\alpha y + z_1).$$

Introduce z by

$$z_1 = \frac{y^2}{\kappa_1 q z}, \qquad z_2 = \kappa_1\kappa_2 q z.$$

Then the matrix $B_0(t) = (B_{ij})$ is parametrized as follows:

$$B_{11} = qy,$$
$$B_{12} = qw\bar{z},$$
$$B_{21} = \frac{-\kappa_1 q y}{w(1-\kappa_1 q\bar{z})}\left(-\bar{\alpha} - \frac{\bar{y}}{\kappa_1 q\bar{z}}\right),$$
$$B_{22} = \frac{-\kappa_1 q\bar{z}}{1-\kappa_1 q\bar{z}}\left(-\bar{\alpha} - \frac{\bar{y}}{\kappa_1 q\bar{z}}\right).$$

Set further

$$b_1 = \frac{1}{\theta_1}, \qquad b_2 = -\frac{\theta_1}{\kappa_1\kappa_2}, \qquad b_3 = \frac{1}{\kappa_1 q}, \qquad a_4 = -\kappa_2. \tag{96}$$

The compatibility (90) is equivalent to

$$\frac{y\bar{y}}{a_4} = -\frac{\bar{z}(\bar{z} - b_2 t)}{\bar{z} - b_3}, \tag{97}$$

$$\frac{z\bar{z}}{b_3} = \frac{y^2}{a_4}, \tag{98}$$

$$\frac{\bar{w}}{w} = -\frac{\bar{z}}{b_3} + 1. \tag{99}$$

We have a constraint

$$\frac{b_1 b_2}{b_3} = q\frac{1}{a_4}.$$

q-$P(A_7')$ is (97) and (98).

3. Degenerations

Some replacements of the parameters for the degenerations of the q-Painlevé equations were given in the paper, [6], for example. In this section, we present replacements of the parameters of the Lax formalisms.

Replace in q-$P(A_3)$, t by εt, y by εy, z by εz, a_3 by εa_3, a_4 by $\varepsilon^{-1}a_4 b_3$ by εb_3 and b_4 by ε^{-1} and let ε tend to zero. Then we obtain q-$P(A_4)$,

$$\frac{y\bar{y}}{a_3 a_4} = -\frac{(\bar{z} - b_1 t)(\bar{z} - b_2 t)}{\bar{z} - b_3},$$
$$\frac{z\bar{z}}{b_3} = -\frac{(y - a_1 t)(y - a_2 t)}{a_4(y - a_3)}.$$

For the sake of simplification of notation, the replacement and the succeeding limiting process will be written as follows:

$$t \to \varepsilon t \qquad y \to \varepsilon y, \qquad z \to \varepsilon z,$$
$$a_3 \to \varepsilon a_3, \qquad a_4 \to \varepsilon^{-1} a_4 \qquad b_3 \to \varepsilon b_3, \qquad b_4 \to \varepsilon^{-1}.$$

By the use of notation as above, the degeneration from the Lax form of q-$P(A_3)$ to that of q-$P(A_4)$ is given by the following scheme:

q-$P(A_3)$ to q-$P(A_4)$:

$$t \to \varepsilon t \qquad y \to \varepsilon y, \qquad z \to \varepsilon z,$$
$$a_3 \to \varepsilon a_3, \qquad a_4 \to \varepsilon^{-1} a_4 \qquad b_3 \to \varepsilon b_3, \qquad b_4 \to \varepsilon^{-1},$$
$$x \to \varepsilon x, \qquad z_1 \to \varepsilon^2 z_1, \qquad w \to \varepsilon^{-1} w, \qquad \kappa_1 \to \varepsilon^{-1}\kappa_1, \qquad \kappa_2 \to \varepsilon,$$
$$\alpha \to \varepsilon\alpha, \qquad \beta \to \varepsilon^{-1}\beta, \qquad \delta \to \varepsilon\delta,$$
$$Y(x,t) \to x^{\log_q \varepsilon} Y(x,t), \qquad A(x,t) \to \varepsilon A(x,t),$$
$$A_0(t) \to \varepsilon A_0(t), \qquad A_2 \to \varepsilon^{-1} A_2, \qquad B_0(t) \to \varepsilon B_0(t),$$
$$B_{11} \to \varepsilon B_{11}, \qquad B_{12} \to \varepsilon B_{12}, \qquad B_{21} \to \varepsilon B_{21}, \qquad B_{22} \to \varepsilon B_{22}.$$

q-$P(A_4)$ to q-$P(A_5)$:

$$a_3 \to \varepsilon, \qquad b_2 \to \varepsilon^{-1} b_2, \qquad \theta_2 \to \varepsilon.$$

q-$P(A_4)$ to q-$P(A_5)^\sharp$:

$$t \to \varepsilon t, \qquad a_1 \to \varepsilon^{-1} a_1, \qquad a_2 \to \varepsilon, \qquad b_1 \to \varepsilon b_1, \qquad b_2 \to \varepsilon^{-1} b_2,$$
$$\theta_1 \to \varepsilon^{-1}\theta_1, \qquad \theta_2 \to \varepsilon.$$

q-$P(A_5)$ to q-$P(A_6)$:

$$t \to \varepsilon t, \qquad a_1 \to \varepsilon^{-1} a_1, \qquad a_2 \to \varepsilon,$$
$$b_1 \to \varepsilon b_1, \qquad b_2 \to \varepsilon^{-1} b_2, \qquad \theta_1 \to \varepsilon^{-1}\theta_1.$$

q-$P(A_5)^\sharp$ to q-$P(A_6)$:

$$a_3 \to \varepsilon, \qquad b_2 \to \varepsilon^{-1} b_2.$$

q-$P(A_5)^\sharp$ to q-$P(A_6)^\sharp$:

$$a_1 \to \varepsilon, \qquad b_1 \to \varepsilon b_1.$$

q-$P(A_6)$ to q-$P(A_7)$:

$$a_1 \to \varepsilon, \qquad b_1 \to \varepsilon b_1.$$

q-$P(A_6)^\sharp$ to q-$P(A_7)$:

$$a_3 \to \varepsilon, \qquad b_2 \to \varepsilon^{-1} b_2.$$

q-$P(A_6)^\sharp$ to q-$P(A_7')$:

$$a_3 \to \varepsilon^{-1}, \qquad a_4 \to \varepsilon a_4, \qquad \kappa_2 \to \varepsilon \kappa_2.$$

4. Degeneration from q-$P(A_2)$ to q-$P(A_3)$

The Lax form of q-$P(A_2)$ was given in the Sakai's paper, [9]. This Lax form yields the Lax form of q-$P(A_3)$ by a process of coalescence.

4.1. Lax form of q-$P(A_2)$

In this subsection, we illustrate the Lax form of q-$P(A_2)$ in the paper, [9].

A Lax pair for the q-Painlevé equation of type A_2 is a linear probrem of the form

$$Y(qx,t) = A(x,t)Y(x,t), \tag{100}$$

$$Y(x,qt) = B(x,t)Y(x,t), \tag{101}$$

whose compatibility condition, namely,

$$A(x,qt)B(x,t) = B(qx,t)A(x,t) \tag{102}$$

can express q-$P(A_2)$. We take $A(x,t)$ to be of the form

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2(t) + x^3A_3, \tag{103}$$

$$A_3 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1 t, \theta_2 t, \tag{104}$$

$$\det A(x,t) = \kappa_1\kappa_2(x-a_1)(x-a_2)(x-a_3)(x-a_4)(x-a_5t)(x-a_6t). \tag{105}$$

We have

$$\kappa_1\kappa_2 a_1 a_2 a_3 a_4 a_5 a_6 = \theta_1\theta_2.$$

The matrix $B(x,t)$ is a rational function of the form

$$B(x,t) = \frac{x}{(x-a_5qt)(x-a_6qt)}(xI + B_0(t)). \tag{106}$$

Define $\lambda = \lambda(t)$, $\mu = \mu(t)$ and $\tilde{\mu} = \tilde{\mu}(t)$ by

$$A_{12}(\lambda,t) = 0, \qquad A_{11}(\lambda,t) = \kappa_1\tilde{\mu}, \qquad A_{22}(\lambda,t) = \kappa_2\mu, \tag{107}$$

so that

$$\mu\tilde{\mu} = \kappa_1\kappa_2(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)(\lambda-a_5t)(\lambda-a_6t).$$

The matrix $A(x,t)$ can be parametrized as

$$A(x,t) = \begin{pmatrix} \kappa_1 W(x,t) & \kappa_2 w L(x,t) \\ \kappa_1 w^{-1} X(x,t) & \kappa_2 Z(x,t) \end{pmatrix},$$

Here

$$L(x,t) = x - \lambda,$$
$$Z(x,t) = (x-\lambda)(x^2 + (\gamma+\lambda)x + \delta) + \mu,$$
$$W(x,t) = (x-\lambda)(x^2 + (-\gamma+\lambda-\sigma_1)x + \tilde{\delta}) + \tilde{\mu},$$
$$X(x,t) = \frac{W(x)Z(x) - \prod_{i=1}^{6}(x-a_i)}{L(x)},$$
$$\delta = \frac{1}{\kappa_1-\kappa_2}\left[\kappa_1(2\lambda^2 - \sigma_1\lambda + \sigma_2 + \gamma(\gamma+\sigma_1)) - \frac{1}{\lambda}(\kappa_1\tilde{\mu} + \kappa_2\mu - \theta_1 - \theta_2)\right],$$
$$\tilde{\delta} = \frac{1}{\kappa_1-\kappa_2}\left[-\kappa_2(2\lambda^2 - \sigma_1\lambda + \sigma_2 + \gamma(\gamma+\sigma_1)) + \frac{1}{\lambda}(\kappa_1\tilde{\mu} + \kappa_2\mu - \theta_1 - \theta_2)\right],$$
$$\tilde{\mu} = \frac{1}{\mu}\prod_{i=1}^{6}(\lambda - a_i), \qquad \sigma_1 = \sum_{i=1}^{6} a_i, \qquad \sigma_2 = \sum_{i<j} a_i a_j.$$

If $q\kappa_1 = \kappa_2$, then the compatibility (102) is equivalent to

$$(\lambda - \underline{\nu})(\lambda - \nu) = \frac{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)}{(\lambda-a_5t)(\lambda-a_6t)}, \tag{108}$$

$$\left(1 - \frac{\nu}{\bar{\lambda}}\right)\left(1 - \frac{\nu}{\lambda}\right) = \frac{a_5a_6}{q}\frac{(\nu-a_1)(\nu-a_2)(\nu-a_3)(\nu-a_4)}{(a_5a_6t + \theta_1/\kappa_2)(a_5a_6t + \theta_2/\kappa_2)}, \tag{109}$$

$$a_5a_6t\lambda\bar{\lambda}(a_1+a_2+a_3+a_4+\bar{\gamma}-\nu)((a_5+a_6)t+\gamma+\nu)$$
$$+ q(a_5a_6t\nu + \theta_1/\kappa_2)(a_5a_6t\nu + \theta_2/\kappa_2) = 0. \tag{110}$$

q-$P(A_2)$ is (108) and (109).

4.2. Degeneration

In this subsection, we give replacements of the parameters for the degeneration. By the use of notation in the section 3, the degeneration from the Lax form of q-$P(A_2)$ to that of q-$P(A_3)$ is given by the following scheme:

$$\lambda \to \varepsilon y, \qquad \nu \to \varepsilon^{-1}z,$$
$$a_1 \to \varepsilon a_3, \qquad a_2 \to \varepsilon a_4, \qquad a_3 \to -\varepsilon^{-1},$$
$$a_4 \to -\varepsilon^{-1}\kappa_1\kappa_2^{-1}q, \qquad a_5 \to \varepsilon a_1, \qquad a_6 \to \varepsilon a_2,$$
$$x \to \varepsilon x, \qquad \mu \to \varepsilon z_2, \qquad \tilde{\mu} \to \varepsilon\kappa_1\kappa_2{}^{-1}qz_1, \qquad \kappa_1 \to \varepsilon^{-1}\kappa_2 q^{-1}, \qquad \kappa_2 \to \varepsilon^{-1}\kappa_2,$$
$$\gamma \to \varepsilon^{-1} + \varepsilon\gamma_1 + O(\varepsilon^2),$$
$$\gamma_1 = \frac{1}{\kappa_1-\kappa_2}[y^{-1}((\theta_1+\theta_2)t - \kappa_1z_1 - \kappa_2z_2) - \kappa_2((a_1+a_2)t + a_3 + a_4) + y(\kappa_1+\kappa_2)].$$

5. Concluding remarks

By the limiting procedure, we derived Lax forms of q-Painlevé equations which are obtained from q-$P(A_3)$. The degeneration scheme from the Lax form of q-$P(A_2)$ was also given. However Lax forms of q-$P(A_0^*)$ and q-$P(A_1)$ do not appear today, the full degeneration pattern cannot be presented. An interesting future problem is to find the relation between the Lax pairs in the paper, [3], and our result.

Acknowledgments

The author expresses his sincere gratitude to Professor Hidetaka Sakai, who gave suggestions about ideas on this research.

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